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WIENER RANDOMIZATION ON UNBOUNDED DOMAINS AND AN APPLICATION TO ALMOST SURE WELL-POSEDNESS OF NLS

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Abstract. We consider a randomization of a function on $\mathbb{R}^d$ that is naturally associated to the Wiener decomposition and, intrinsically, to the modulation spaces. Such randomized functions enjoy better integrability, thus allowing us to improve the Strichartz estimates for the Schrödinger equation. As an example, we also show that the energy-critical cubic nonlinear Schrödinger equation on $\mathbb{R}^4$ is almost surely locally well-posed with respect to randomized initial data below the energy space.

1. Introduction

1.1. **Background.** The Cauchy problem of the nonlinear Schrödinger equation (NLS):

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1}u \\ u\big|_{t=0} = u_0 \in H^s(\mathbb{R}^d), \end{cases} \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^d \tag{1.1}$$

has been studied extensively over recent years. One of the key ingredients in studying (1.1) is the dispersive effect of the associated linear flow. Such dispersion is often expressed in terms of the Strichartz estimates (see Lemma 1.2 below), which have played an important role in studying various problems on (1.1), in particular, local and global well-posedness issues.

It is well-known that (1.1) is invariant under several symmetries. In the following, we concentrate on the dilation symmetry. The dilation symmetry states that if $u(t,x)$ is a solution to (1.1) on $\mathbb{R}^d$ with an initial condition u_0, then $u^\lambda(t,x) = \lambda^{-\frac{2}{p-1}} u(\lambda^{-2}t, \lambda^{-1}x)$ is also a solution to (1.1) with the λ-scaled initial condition $u_0^\lambda(x) = \lambda^{-\frac{2}{p-1}} u_0(\lambda^{-1}x)$. Associated to the dilation symmetry, there is a scaling-critical Sobolev index $s_c := \frac{d}{2} - \frac{2}{p-1}$ such that the homogeneous $\dot{H}^{s_c}$-norm is invariant under the dilation symmetry. For example, when $p = \frac{4}{d-2} + 1$, we have $s_c = 1$ and (1.1) is called energy-critical. It is known that (1.1) is ill-posed in the supercritical regime, that is, in H^s for $s < s_c$; see [19, 12, 17, 1].

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In an effort to study the invariance of the Gibbs measure for the defocusing (Wick ordered) cubic NLS on $\mathbb{T}^2$, Bourgain [7] considered random initial data of the form:

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{in\cdot x}, \tag{1.2}$$

where $\{g_n\}_{n\in\mathbb{Z}^2}$ is a sequence of independent complex-valued standard Gaussian random variables. The function (1.2) represents a typical element in the support of the Gibbs measure, more precisely, in the support of the Gaussian free field on $\mathbb{T}^2$ associated to this Gibbs measure, and is critical with respect to the scaling. With a combination of deterministic PDE techniques and probabilistic arguments, Bourgain showed that the (Wick ordered) cubic NLS on $\mathbb{T}^2$ is well-posed almost surely with respect to random initial data (1.2). Burq-Tzvetkov [15] further explored the study of Cauchy problems with more general random initial data. They considered the cubic nonlinear wave equation (NLW) on a three dimensional compact Riemannian manifold M without a boundary, where the scaling-critical Sobolev index s_c is given by $s_c = \frac{1}{2}$. Given $u_0(x) = \sum_{n=1}^\infty c_n e_n(x) \in H^s(M)$, $s \geq \frac{1}{4}$, they proved almost sure local well-posedness with random initial data of the form:[1]

$$u_0^\omega(x) = \sum_{n=1}^\infty g_n(\omega) c_n e_n(x) \tag{1.3}$$

where $\{g_n\}_{n=1}^\infty$ is a sequence of independent mean-zero random variables with a uniform bound on the fourth moments and $\{e_n\}_{n=1}^\infty$ is an orthonormal basis of $L^2(M)$ consisting of the eigenfunctions of the Laplace-Beltrami operator. It was also shown that u_0^ω in (1.3) has the same Sobolev regularity as the original function u_0 and is not smoother, almost surely. In particular, if $u_0 \in H^s(M) \setminus H^{\frac{1}{2}}(M)$, their result implies almost sure local well-posedness in the supercritical regime. There are several works on Cauchy problems of evolution equations with random data that followed these results, including some on almost sure global well-posedness: [8, 50, 20, 13, 21, 22, 16, 38, 23, 45, 14, 10, 11, 39, 44, 37].

We point out that many of these works are on compact domains, where there is a countable basis of eigenfunctions of the Laplacian and thus there is a natural way to introduce a randomization. On $\mathbb{R}^d$, randomizations were introduced with respect to a countable basis of eigenfunctions of the Laplacian with a confining potential such as a harmonic oscillator $\Delta - |x|^2$; we note that functions in Sobolev spaces associated to the Laplacian with a confining potential have an extra decay in space. Our goal is to introduce a randomization for functions in the usual Sobolev spaces on $\mathbb{R}^d$ without such extra decay. For this purpose, we first review some basic notions and facts concerning the so-called *modulation spaces* of time-frequency analysis.

1.2. **Modulation spaces.** The modulation spaces were introduced by Feichtinger [24] in early eighties. In following collaborations with Gröchenig [25, 26], they established the basic theory of these function spaces, in particular their invariance, continuity, embeddings, and convolution properties. The difference between the Besov spaces and the modulation spaces

[1]For NLW, one needs to specify $(u, \partial_t u)|_{t=0}$ as an initial condition. For simplicity of presentation, we only displayed $u|_{t=0}$ in (1.3).

consists in the geometry of the frequency space employed: the dyadic annuli in the definition of the former spaces are replaced by unit cubes Q_n centered at $n \in \mathbb{Z}^d$ in the definition of the latter ones. Thus, the modulation spaces arise via a uniform partition of the frequency space $\mathbb{R}^d = \bigcup_{n\in\mathbb{Z}^d} Q_n$, which is commonly referred to as a *Wiener decomposition* [54]. In certain contexts, this decomposition allows for a finer analysis by effectively capturing the time-frequency concentration of a distribution.

For $x, \xi \in \mathbb{R}^d$, let $\mathcal{F}u(\xi) = \widehat{u}(\xi) = \int_{\mathbb{R}^d} u(x)e^{-2\pi i x\cdot\xi}\,dx$ denote the Fourier transform of a distribution u. Typically, the (weighted) modulation spaces $M_s^{p,q}(\mathbb{R}^d)$, $p, q > 0, s \in \mathbb{R}$, are defined by imposing the $L^p(dx)L^q(\langle\xi\rangle^s d\xi)$ integrability of the short-time (or windowed) Fourier transform of a distribution $V_\phi u(x,\xi) := \mathcal{F}(u\overline{T_x\phi})(\xi)$. Here, $\langle\xi\rangle^s = (1+|\xi|^2)^{\frac{s}{2}}$, ϕ is some fixed non-zero Schwartz function, and T_x denotes the translation defined by $T_x(\phi)(y) = \phi(y-x)$. When $s = 0$, one simply writes $M^{p,q}$. Modulation spaces satisfy some desirable properties: they are quasi-Banach spaces, two different windows ϕ_1, ϕ_2 yield equivalent norms, $M_s^{2,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$, $(M_s^{p,q}(\mathbb{R}^d))' = M_{-s}^{p',q'}(\mathbb{R}^d)$, $M_{s_1}^{p_1,q_1}(\mathbb{R}^d) \subset M_{s_2}^{p_2,q_2}(\mathbb{R}^d)$ for $s_1 \geq s_2$, $p_1 \leq p_2$, and $q_1 \leq q_2$, and $\mathcal{S}(\mathbb{R}^d)$ is dense in $M_s^{p,q}(\mathbb{R}^d)$.

We prefer to use an equivalent norm on the modulation space $M_s^{p,q}$, which is induced by a corresponding Wiener decomposition of the frequency space. Given $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\operatorname{supp}\psi \subset [-1,1]^d$ and $\sum_{n\in\mathbb{Z}^d}\psi(\xi - n) \equiv 1$, let

$$\|u\|_{M_s^{p,q}(\mathbb{R}^d)} = \big\|\langle n\rangle^s \|\psi(D-n)u\|_{L_x^p(\mathbb{R}^d)}\big\|_{\ell_n^q(\mathbb{Z}^d)}. \tag{1.4}$$

Note that $\psi(D-n)$ is just a Fourier multiplier with symbol χ_{Q_n} conveniently smoothed:

$$\psi(D-n)u(x) = \int_{\mathbb{R}^d} \psi(\xi - n)\widehat{u}(\xi)e^{2\pi i x\cdot\xi}\,d\xi.$$

It is worthwhile to compare the definition (1.4) with the one for the Besov spaces which uses a dyadic partition of the frequency domain. Let $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\operatorname{supp}\varphi_0 \subset \{|\xi| \leq 2\}$, $\operatorname{supp}\varphi \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$, and $\varphi_0(\xi) + \sum_{j=1}^\infty \varphi(2^{-j}\xi) \equiv 1$. With $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, we define the Besov spaces $B_s^{p,q}$ via the norm

$$\|u\|_{B_s^{p,q}(\mathbb{R}^d)} = \big\|2^{js}\|\varphi_j(D)u\|_{L^p(\mathbb{R}^d)}\big\|_{\ell_j^q(\mathbb{Z}_{\geq 0})}. \tag{1.5}$$

There are several known embeddings between the Besov, Sobolev, and modulation spaces; see, for example, Okoudjou [40], Toft [51], Sugimoto-Tomita [48], and Kobayashi-Sugimoto [35].

1.3. **Randomization adapted to the Wiener decomposition.** Given a function ϕ on $\mathbb{R}^d$, we have

$$\phi = \sum_{n\in\mathbb{Z}^d} \psi(D-n)\phi,$$

where $\psi(D-n)$ is defined above. The identity above leads to a randomization that is naturally associated to the Wiener decomposition, and hence to the modulation spaces, as follows. Let $\{g_n\}_{n\in\mathbb{Z}^d}$ be a sequence of independent mean zero complex-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$, where the real and imaginary parts of g_n are independent and endowed with probability distributions $\mu_n^{(1)}$ and $\mu_n^{(2)}$. Then, we can define the *Wiener randomization of* ϕ by

$$\phi^\omega := \sum_{n\in\mathbb{Z}^d} g_n(\omega)\psi(D-n)\phi. \tag{1.6}$$

We note that Lührmann-Mendelson [37] also considered a similar randomization of the form (1.6) in the study of NLW on $\mathbb{R}^3$. See Remark 1.10 below. The randomization in [37] stems from yet another one used by Zhang and Fang [56] in their study of the Navier-Stokes equations. We point out, however, that the main purpose of our paper is to explain how the randomization of the form (1.6) is naturally associated to the Wiener decomposition and hence the modulation spaces. See also our previous paper [3] in the periodic setting. Thus, from the perspective of time-frequency analysis, the Wiener randomization seems to be the "right" one.

In the sequel, we make the following assumption; there exists $c > 0$ such that

$$\left| \int_{\mathbb{R}} e^{\gamma x} d\mu_n^{(j)}(x) \right| \le e^{c\gamma^2} \tag{1.7}$$

for all $\gamma \in \mathbb{R}$, $n \in \mathbb{Z}^d$, $j = 1, 2$. Note that (1.7) is satisfied by standard complex-valued Gaussian random variables, standard Bernoulli random variables, and any random variables with compactly supported distributions.

It is easy to see that, if $\phi \in H^s(\mathbb{R}^d)$, then the randomized function ϕ^ω is almost surely in $H^s(\mathbb{R}^d)$; see Lemma 2.2 below. One can also show that there is no smoothing upon randomization in terms of differentiability; see, for example, Lemma B.1 in [15]. Instead, the main point of this randomization is its improved integrability; if $\phi \in L^2(\mathbb{R}^d)$, then the randomized function ϕ^ω is almost surely in $L^p(\mathbb{R}^d)$ for any finite $p \ge 2$; see Lemma 2.3 below. Such results for random Fourier series are known as Paley-Zygmund's theorem [42]; see also Kahane's book [31] and Ayache-Tzvetkov [2].

Remark 1.1. One may fancy a randomization associated to Besov spaces, of the form:

$$\phi^\omega := \sum_{j=0}^{\infty} g_n(\omega)\varphi(D)\phi.$$

In view of the Littlewood-Paley theory, such a randomization does not yield any improvement on differentiability or integrability and thus it is of no interest.

1.4. **Main results.** The Wiener randomization of an initial condition allows us to establish some improvements of the Strichartz estimates. In turn, these probabilistic Strichartz estimates yield an almost sure well-posedness result for NLS. First, we recall the usual Strichartz estimates on $\mathbb{R}^d$ for the reader's convenience. We say that a pair (q, r) is *Schrödinger admissible* if it satisfies

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \tag{1.8}$$

with $2 \le q, r \le \infty$ and $(q, r, d) \ne (2, \infty, 2)$. Let $S(t) = e^{it\Delta}$. Then, the following Strichartz estimates are known to hold.

Lemma 1.2 ([47, 55, 27, 32]). *Let (q,r) be Schrödinger admissible. Then, we have*

$$\|S(t)\phi\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|\phi\|_{L^2_x(\mathbb{R}^d)}. \tag{1.9}$$

Next, we present improvements of the Strichartz estimates under the Wiener randomization. Proposition 1.3 will be then used for a local-in-time theory, while Proposition 1.4 is useful for small data global theory. The proofs of Propositions 1.3 and 1.4 are presented in Section 2.

Proposition 1.3 (Improved local-in-time Strichartz estimate). *Given $\phi \in L^2(\mathbb{R}^d)$, let ϕ^ω be its randomization defined in* (1.6)*, satisfying* (1.7)*. Then, given $2 \le q, r < \infty$, there exist $C, c > 0$ such that*

$$P\Big(\|S(t)\phi^\omega\|_{L^q_t L^r_x([0,T]\times\mathbb{R}^d)} > \lambda\Big) \le C\exp\bigg(-c\frac{\lambda^2}{T^{\frac{2}{q}}\|\phi\|^2_{L^2}}\bigg) \tag{1.10}$$

for all $T > 0$ and $\lambda > 0$.

In particular, by setting $\lambda = T^\theta\|\phi\|_{L^2}$, we have

$$\|S(t)\phi^\omega\|_{L^q_t L^r_x([0,T]\times\mathbb{R}^d)} \lesssim T^\theta\|\phi\|_{L^2(\mathbb{R}^d)}$$

outside a set of probability at most $C\exp\big(-cT^{2\theta-\frac{2}{q}}\big)$. Note that, as long as $\theta < \frac{1}{q}$, this probability can be made arbitrarily small by letting $T \to 0$. Moreover, for fixed $T > 0$, we have the following; given any small $\varepsilon > 0$, we have

$$\|S(t)\phi^\omega\|_{L^q_t L^r_x([0,T]\times\mathbb{R}^d)} \le C_T\Big(\log\frac{1}{\varepsilon}\Big)^{\frac{1}{2}}\|\phi\|_{L^2}$$

outside a set of probability $< \varepsilon$.

The next proposition states an improvement of the Strichartz estimates in the global-in-time setting.

Proposition 1.4 (Improved global-in-time Strichartz estimate). *Given $\phi \in L^2(\mathbb{R}^d)$, let ϕ^ω be its randomization defined in* (1.6)*, satisfying* (1.7)*. Given a Schrödinger admissible pair (q,r) with $q, r < \infty$, let $\widetilde{r} \ge r$. Then, there exist $C, c > 0$ such that*

$$P\Big(\|S(t)\phi^\omega\|_{L^q_t L^{\widetilde{r}}_x(\mathbb{R}\times\mathbb{R}^d)} > \lambda\Big) \le Ce^{-c\lambda^2\|\phi\|^{-2}_{L^2}} \tag{1.11}$$

for all $\lambda > 0$. In particular, given any small $\varepsilon > 0$, we have

$$\|S(t)\phi^\omega\|_{L^q_t L^{\widetilde{r}}_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \Big(\log\frac{1}{\varepsilon}\Big)^{\frac{1}{2}}\|\phi\|_{L^2}$$

outside a set of probability at most ε.

We conclude this introduction by discussing an example of almost sure local well-posedness of NLS with randomized initial data below a scaling critical regularity. In the following, we consider the energy-critical cubic NLS on $\mathbb{R}^4$:

$$i\partial_t u + \Delta u = \pm|u|^2 u, \qquad (t,x) \in \mathbb{R}\times\mathbb{R}^4. \tag{1.12}$$

Cazenave-Weissler [18] proved local well-posedness of (1.12) with initial data in the critical space $\dot{H}^1(\mathbb{R})$. See Ryckman-Vişan [46], Vişan [52], and Kenig-Merle [33] for global-in-time results. In the following, we state a local well-posedness result of (1.12) with random initial data below the critical space. More precisely, given $\phi \in H^s(\mathbb{R}^4) \setminus H^1(\mathbb{R}^4)$, $s \in (\frac{3}{5}, 1)$, and its randomization ϕ^ω defined in (1.6), we prove that (1.12) is almost surely locally well-posed with random initial data ϕ^ω. Although ϕ and its randomization ϕ^ω lie in a supercritical regularity regime, the Wiener randomization essentially makes the problem *subcritical*. This is a common feature for many of the probabilistic well-posedness results.

Theorem 1.5. *Let $s \in (\frac{3}{5}, 1)$. Given $\phi \in H^s(\mathbb{R}^4)$, let ϕ^ω be its randomization defined in* (1.6)*, satisfying* (1.7)*. Then, the cubic NLS* (1.12) *on $\mathbb{R}^4$ is almost surely locally well-posed with respect to the randomization ϕ^ω as initial data. More precisely, there exist $C, c, \gamma > 0$ and $\sigma = 1+$ such that for each $T \ll 1$, there exists a set $\Omega_T \subset \Omega$ with the following properties:*

(i) $P(\Omega \setminus \Omega_T) \leq C \exp\Big(- \frac{c}{T^\gamma \|\phi\|_{H^s}^2}\Big)$,

(ii) *For each $\omega \in \Omega_T$, there exists a (unique) solution u to* (1.12) *with $u|_{t=0} = \phi^\omega$ in the class*

$$S(t)\phi^\omega + C([-T, T] : H^\sigma(\mathbb{R}^4)) \subset C([-T, T] : H^s(\mathbb{R}^4)).$$

The details of the proof of Theorem 1.5 are presented in Section 3. We discuss here a very brief outline of the argument. Denoting the linear and nonlinear parts of u by $z(t) = z^\omega(t) := S(t)\phi^\omega$ and $v(t) := u(t) - S(t)\phi^\omega$ respectively, we can reduce (1.12) to

$$\begin{cases} i\partial_t v + \Delta v = \pm |v + z|^2 (v + z) \\ v|_{t=0} = 0. \end{cases} \tag{1.13}$$

We then prove that the Cauchy problem (1.13) is almost surely locally well-posed for v, viewing z as a random forcing term. This is done by using the standard subcritical $X^{s,b}$-spaces with $b > \frac{1}{2}$ defined by

$$\|u\|_{X^{s,b}(\mathbb{R} \times \mathbb{R}^4)} = \|\langle \xi \rangle^s \langle \tau + |\xi|^2 \rangle^b \widehat{u}(\tau, \xi)\|_{L^2_{\tau,\xi}(\mathbb{R} \times \mathbb{R}^4)}.$$

We point out that the uniqueness in Theorem 1.5 refers to uniqueness of the nonlinear part $v(t) = u(t) - S(t)\phi^\omega$ of a solution u.

We conclude this introduction with several remarks.

Remark 1.6. Theorem 1.5 holds for both defocusing and focusing cases (corresponding to the $+$ sign and the $-$ sign in (1.1), respectively) due to the local-in-time nature of the problem.

Remark 1.7. Theorem 1.5 can also be proven with the variants of the $X^{s,b}$-spaces adapted to the U^p- and V^p-spaces introduced by Koch, Tataru, and their collaborators [36, 29, 30]. These spaces are designed to handle problems in critical regularities. We decided to present the proof with the classical subcritical $X^{s,b}$-spaces, $b > \frac{1}{2}$, to emphasize that the problem has become subcritical upon randomization. We should, however, point out that, with the spaces introduced by Koch and Tataru, we can also prove probabilistic small data global well-posedness and scattering as a consequence of the probabilistic global-in-time Strichartz estimates (Proposition 1.4). See our paper [4] for an example of such results for the cubic NLS on $\mathbb{R}^d$, $d \geq 3$.

It is of interest to consider almost sure global existence for (1.12). While the mass of v in (1.13) has a global-in-time control, there is no energy conservation for v and thus we do not know how to proceed at this point. In [4], we establish almost sure global existence for (1.12), assuming an a priori control on the H^1-norm of the nonlinear part v of a solution. We also prove there, without any assumption, global existence with a large probability by considering a randomization, not on unit cubes but on dilated cubes this time.

In the context of the defocusing cubic NLW on $\mathbb{R}^4$, one can obtain an a priori bound on the energy of the nonlinear part of a solution, see [16]. As a consequence, the third author [43] proved almost sure global well-posedness of the energy-critical defocusing cubic NLW on $\mathbb{R}^4$ below the scaling critical regularity.

Remark 1.8. In Theorem 1.5, we simply used $\sigma = 1+$ as the regularity of the nonlinear part v. It is possible to characterize the possible values of σ in terms of the regularity $s < 1$ of ϕ. However, for simplicity of presentation, we omitted such a discussion.

Remark 1.9. In probabilistic well-posedness results [6, 8, 20, 39] for NLS on $\mathbb{T}^d$, random initial data are assumed to be of the following specific form:

$$u_0^\omega(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{(1 + |n|^2)^{\frac{\alpha}{2}}} e^{in\cdot x}, \tag{1.14}$$

where $\{g_n\}_{n\in\mathbb{Z}^d}$ is a sequence of independent complex-valued standard Gaussian random variables. The expression (1.14) has a close connection to the study of invariant (Gibbs) measures and, hence, it is of importance. At the same time, due to the lack of a full range of Strichartz estimates on $\mathbb{T}^d$, one could not handle a general randomization of a given function as in (1.3). In Theorem 1.5, we consider NLS on $\mathbb{R}^4$ and thus we do not encounter this issue thanks to a full range of the Strichartz estimates. For NLW, finite speed of propagation allows us to use a full range of Strichartz estimates even on compact domains, at least locally in time; thus, in that context, one does not encounter such an issue.

Remark 1.10. In a recent preprint, Lührmann-Mendelson [37] considered the defocusing NLW on $\mathbb{R}^3$ with randomized initial data defined in (1.6) in a supercritical regularity and proved almost sure global well-posedness in the energy-subcritical case, following the method developed in [20]. For the energy-critical quintic NLW, they obtained almost sure local well-posedness along with small data global existence and scattering.

2. Probabilistic Strichartz estimates

In this section, we state and prove some basic properties of the randomized function ϕ^ω defined in (1.6), including the improved Strichartz estimates (Propositions 1.3 and 1.4). First, recall the following probabilistic estimate. See [15] for the proof.

Lemma 2.1. *Assume* (1.7). *Then, there exists* $C > 0$ *such that*

$$\Big\| \sum_{n\in\mathbb{Z}^d} g_n(\omega) c_n \Big\|_{L^p(\Omega)} \le C\sqrt{p}\|c_n\|_{\ell^2_n(\mathbb{Z}^d)}$$

for all $p \ge 2$ *and* $\{c_n\} \in \ell^2(\mathbb{Z}^d)$.

Given $\phi \in H^s$, it is easy to see that its randomization $\phi^\omega \in H^s$ almost surely, for example, if $\{g_n\}$ has a uniform finite variance. Under the assumption (1.7), we have a more precise description on the size of ϕ^ω.

Lemma 2.2. *Given* $\phi \in H^s(\mathbb{R}^d)$, *let* ϕ^ω *be its randomization defined in* (1.6), *satisfying* (1.7). *Then, we have*

$$P\Big(\|\phi^\omega\|_{H^s(\mathbb{R}^d)} > \lambda \Big) \le Ce^{-c\lambda^2 \|\phi\|^{-2}_{H^s}} \tag{2.1}$$

for all $\lambda > 0$.

Proof. By Minkowski's integral inequality and Lemma 2.1, we have

$$\begin{aligned}\Big(\mathbb{E}\|\phi^\omega\|^p_{H^s(\mathbb{R}^d)} \Big)^{\frac{1}{p}} &\le \big\| \|\langle \nabla \rangle^s \phi^\omega\|_{L^p(\Omega)} \big\|_{L^2_x(\mathbb{R}^d)} \lesssim \sqrt{p} \big\| \|\psi(D-n)\langle \nabla\rangle^s \phi\|_{\ell^2_n} \big\|_{L^2_x} \\ &\sim \sqrt{p}\|\phi\|_{H^s}\end{aligned}$$

for any $p \ge 2$. Thus, we have obtained

$$\mathbb{E}[\|\phi^\omega\|^p_{H^s}] \le C_0^p p^{\frac{p}{2}} \|\phi\|^p_{H^s}.$$

By Chebyshev's inequality, we have

$$P\Big(\|\phi^\omega\|_{H^s} > \lambda \Big) < \bigg(\frac{C_0 p^{\frac{1}{2}} \|\phi\|_{H^s}}{\lambda} \bigg)^p \tag{2.2}$$

for $p \ge 2$.

Let $p = \Big(\frac{\lambda}{C_0 e \|\phi\|_{H^s}} \Big)^2$. If $p \ge 2$, then by (2.2), we have

$$P\Big(\|\phi^\omega\|_{H^s} > \lambda \Big) < \bigg(\frac{C_0 p^{\frac{1}{2}} \|\phi\|_{H^s}}{\lambda} \bigg)^p = e^{-p} = e^{-c\lambda^2\|\phi\|^{-2}_{H^s}}.$$

Otherwise, i.e. if $p = \Big(\frac{\lambda}{C_0 e \|\phi\|_{H^s}} \Big)^2 \le 2$, we can choose C such that $Ce^{-2} \ge 1$. Then, we have

$$P\Big(\|\phi^\omega\|_{H^s} > \lambda \Big) \le 1 \le Ce^{-2} \le Ce^{-c\lambda^2\|\phi\|^{-2}_{H^s}},$$

thus giving the desired result. □

The next lemma shows that, if $\phi \in L^2(\mathbb{R}^d)$, then its randomization ϕ^ω is almost surely in $L^p(\mathbb{R}^d)$ for any $p \in [2, \infty)$.

Lemma 2.3. *Given* $\phi \in L^2(\mathbb{R}^d)$, *let* ϕ^ω *be its randomization defined in* (1.6), *satisfying* (1.7). *Then, given finite* $p \ge 2$, *there exist* $C, c > 0$ *such that*

$$P\Big(\|\phi^\omega\|_{L^p(\mathbb{R}^d)} > \lambda \Big) \le Ce^{-c\lambda^2 \|\phi\|^{-2}_{L^2}} \tag{2.3}$$

for all $\lambda > 0$. *In particular,* ϕ^ω *is in* $L^p(\mathbb{R}^d)$ *almost surely.*

Proof. By Lemma 2.1, we have

$$\begin{aligned}\Big(\mathbb{E}\|\phi^\omega\|^r_{L^p_x(\mathbb{R}^d)}\Big)^{\frac{1}{r}} &\le \big\|\|\phi^\omega\|_{L^r(\Omega)}\big\|_{L^p_x(\mathbb{R}^d)} \lesssim \sqrt{r}\big\|\|\psi(D-n)\phi\|_{\ell^2_n}\big\|_{L^p_x}\\ &\le \sqrt{r}\big\|\|\psi(D-n)\phi\|_{L^p_x}\big\|_{\ell^2_n} \le \sqrt{r}\big\|\|\psi(D-n)\phi\|_{L^2_x}\big\|_{\ell^2_n}\\ &\sim \sqrt{r}\|\phi\|_{L^2_x}\end{aligned}$$

for any $r \ge p$. Then, (2.3) follows as in the proof of Lemma 2.2. □

We conclude this section by presenting the proofs of the improved Strichartz estimates under randomization. Before continuing further, we briefly recall the definitions of the smooth projections from Littlewood-Paley theory. Let φ be a smooth real-valued bump function supported on $\{\xi \in \mathbb{R}^d : |\xi| \le 2\}$ and $\varphi \equiv 1$ on $\{\xi : |\xi| \le 1\}$. If $N > 1$ is a dyadic number, we define the smooth projection $\mathbf{P}_{\le N}$ onto frequencies $\{|\xi| \le N\}$ by

$$\widehat{\mathbf{P}_{\le N} f}(\xi) := \varphi\big(\tfrac{\xi}{N}\big)\widehat{f}(\xi).$$

Similarly, we can define the smooth projection $\mathbf{P}_N$ onto frequencies $\{|\xi| \sim N\}$ by

$$\widehat{\mathbf{P}_N f}(\xi) := \Big(\varphi\big(\tfrac{\xi}{N}\big) - \varphi\big(\tfrac{2\xi}{N}\big)\Big)\widehat{f}(\xi).$$

We make the convention that $\mathbf{P}_{\le 1} = \mathbf{P}_1$. Bernstein's inequality states that

$$\|\mathbf{P}_{\le N} f\|_{L^q(\mathbb{R}^d)} \lesssim N^{\frac{d}{p}-\frac{d}{q}}\|\mathbf{P}_{\le N} f\|_{L^p(\mathbb{R}^d)}, \qquad 1 \le p \le q \le \infty. \tag{2.4}$$

The same inequality holds if we replace $\mathbf{P}_{\le N}$ by $\mathbf{P}_N$. As an immediate corollary, we have

$$\|\psi(D-n)\phi\|_{L^q(\mathbb{R}^d)} \lesssim \|\psi(D-n)\phi\|_{L^p(\mathbb{R}^d)}, \qquad 1 \le p \le q \le \infty, \tag{2.5}$$

for all $n \in \mathbb{Z}^d$. This follows from applying (2.4) to $\phi_n(x) := e^{2\pi i n\cdot x}\psi(D-n)\phi(x)$ and noting that $\operatorname{supp}\widehat{\phi}_n \subset [-1,1]^d$. The point of (2.5) is that once a function is (roughly) restricted to a cube, we do not need to lose any regularity to go from the L^q-norm to the L^p-norm, $q \ge p$.

Proof of Proposition 1.3. Let $q, r \ge 2$. We write L^q_T to denote $L^q_t([0,T])$. By Lemma 2.1 and (2.5), we have

$$\begin{aligned}\Big(\mathbb{E}\|S(t)\phi^\omega\|^p_{L^q_tL^r_x([0,T]\times\mathbb{R}^d)}\Big)^{\frac{1}{p}} &\le \Big\|\|S(t)\phi^\omega\|_{L^p(\Omega)}\Big\|_{L^q_TL^r_x} \le \sqrt{p}\Big\|\|\psi(D-n)S(t)\phi\|_{\ell^2_n}\Big\|_{L^q_TL^r_x}\\ &\le \sqrt{p}\Big\|\|\psi(D-n)S(t)\phi\|_{L^r_x}\Big\|_{L^q_T\ell^2_n} \lesssim \sqrt{p}\Big\|\|\psi(D-n)S(t)\phi\|_{L^2_x}\Big\|_{L^q_T\ell^2_n}\\ &\lesssim T^{\frac{1}{q}}\sqrt{p}\|\phi\|_{L^2_x}\end{aligned}$$

for $p \ge \max(q,r)$. Then, (1.10) follows as in the proof of Lemma 2.2. □

Proof of Proposition 1.4. Let (q,r) be Schrödinger admissible and $\widetilde{r} \ge r$. Then, proceeding as before, we have

$$\Big(\mathbb{E}\|S(t)\phi^\omega\|^p_{L^q_tL^{\widetilde{r}}_x(\mathbb{R}\times\mathbb{R}^d)}\Big)^{\frac{1}{p}} \lesssim \sqrt{p}\Big\|\|\psi(D-n)S(t)\phi\|_{L^{\widetilde{r}}_x}\Big\|_{\ell^2_nL^q_t} \lesssim \sqrt{p}\Big\|\|\psi(D-n)S(t)\phi\|_{L^q_tL^r_x}\Big\|_{\ell^2_n}$$

By Lemma 1.2,

$$\lesssim \sqrt{p}\big\|\|\psi(D-n)\phi\|_{L^2_x}\big\|_{\ell^2_n} \sim \sqrt{p}\|\phi\|_{L^2_x}$$

for $p \geq \max(q, \widetilde{r})$. Finally, (1.11) follows as above. □

Remark 2.4. The Cauchy problem (1.1) has also been studied for initial data in the modulation spaces $M^{p,1}_s$ for $1 \leq p \leq \infty$ and $s \geq 0$; see, for example, [5] and [53]. Thus, it is tempting to consider what happens if we randomize an initial condition in a modulation space $M^{p,q}_s$. In this case, however, there is no improvement in the Strichartz estimates in terms of integrability, i.e. p, hence, no improvement of well-posedness with respect to $M^{p,q}_s$ in terms of differentiability, i.e. in s. Indeed, by computing the moments of the modulation norm of the randomized function (1.6), one immediately sees that the modulation norm remains essentially unchanged due to the outside summation over n. In the proof of Propositions 1.3 and 1.4, the averaging effect of a linear combination of the random variables g_n played a crucial role. For the modulation spaces, we do not have such an averaging effect since the outside summation over n forces us to work on a piece restricted to each cube, i.e. each random variable at a time.

3. Almost sure local well-posedness

Given $\phi \in H^s(\mathbb{R}^d)$, let ϕ^ω be its randomization defined in (1.6). In the following, we consider the Cauchy problem (1.12) with random initial data $u|_{t=0} = \phi^\omega$. By letting $z(t) = z^\omega(t) := S(t)\phi^\omega$ and $v(t) := u(t) - S(t)\phi^\omega$, we can reduce (1.1) to

$$\begin{cases} i\partial_t v + \Delta v = \pm|v+z|^2(v+z) \\ v|_{t=0} = 0. \end{cases} \tag{3.1}$$

By expressing (3.1) in the Duhamel formulation, we have

$$v(t) = \mp i \int_0^t S(t-t')\mathcal{N}(v+z)(t')dt', \tag{3.2}$$

where $\mathcal{N}(u) = |u|^2u = u\bar{u}u$. Let η be a smooth cutoff function supported on $[-2,2]$, $\eta \equiv 1$ on $[-1,1]$, and let $\eta_{_T}(t) = \eta\big(\frac{t}{T}\big)$. Note that, if v satisfies

$$v(t) = \mp i\eta(t) \int_0^t S(t-t')\eta_{_T}(t')\mathcal{N}(\eta v + \eta_{_T} z)(t')dt' \tag{3.3}$$

for some $T \ll 1$, then it also satisfies (3.2) on $[-T, T]$. Hence, we consider (3.3) in the following.

Given $z(t) = S(t)\phi^\omega$, define Γ by

$$\Gamma v(t) = \mp i\eta(t) \int_0^t S(t-t')\eta_{_T}(t')\mathcal{N}(\eta v + \eta_{_T} z)(t')dt'. \tag{3.4}$$

Then, the following nonlinear estimates yield Theorem 1.5.

Proposition 3.1. *Let $s \in \big(\frac{3}{5}, 1\big)$. Given $\phi \in H^s(\mathbb{R}^4)$, let ϕ^ω be its randomization defined in* (1.6)*, satisfying* (1.7)*. Then, there exists $\sigma = 1+$, $b = \frac{1}{2}+$ and $\theta = 0+$ such that for each small $T \ll 1$ and $R > 0$, we have*

$$\|\Gamma v\|_{X^{\sigma,b}} \le C_1 T^\theta (\|v\|^3_{X^{\sigma,b}} + R^3), \tag{3.5}$$

$$\|\Gamma v_1 - \Gamma v_2\|_{X^{\sigma,b}} \le C_2 T^\theta (\|v_1\|^2_{X^{\sigma,b}} + \|v_2\|^2_{X^{\sigma,b}} + R^2)\|v_1 - v_2\|_{X^{\sigma,b}}, \tag{3.6}$$

outside a set of probability at most $C \exp\Big(-c\frac{R^2}{\|\phi\|^2_{H^s}}\Big)$.

We first present the proof of Theorem 1.5, assuming Proposition 3.1. Then, we prove Proposition 3.1 at the end of this section.

Proof of Theorem 1.5. Let B_1 denote the ball of radius 1 centered at the origin in $X^{\sigma,b}$. Then, given $T \ll 1$, we show that the map Γ is a contraction on B_1. Given $T \ll 1$, we choose $R = R(T) \sim T^{-\frac{\gamma}{2}}$ for some $\gamma \in (0, \frac{2\theta}{3})$ such that

$$C_1 T^\theta (1 + R^3) \le 1 \quad \text{and} \quad C_2 T^\theta (2 + R^2) \le \frac{1}{2}.$$

Then, for $v, v_1, v_2 \in B_1$, Proposition 3.1 yields

$$\|\Gamma v\|_{X^{\sigma,b}} \le 1,$$
$$\|\Gamma v_1 - \Gamma v_2\|_{X^{\sigma,b}} \le \frac{1}{2}\|v_1 - v_2\|_{X^{\sigma,b}}$$

outside an exceptional set of probability at most

$$C \exp\bigg(-c\frac{R^2}{\|\phi\|^2_{H^s}}\bigg) = C \exp\bigg(-\frac{c}{T^\gamma \|\phi\|^2_{H^s}}\bigg).$$

Therefore, by defining Ω_T to be the complement of this exceptional set, it follows that, for $\omega \in \Omega_T$, there exists a unique $v^\omega \in B_1$ such that $\Gamma v^\omega = v^\omega$. This completes the proof of Theorem 1.5. □

Hence, it remains to prove Proposition 3.1. Before proceeding further, we first present some lemmata on the basic $X^{s,b}$-estimates. See [6, 34, 49] for the basic properties of the $X^{s,b}$-spaces.

Lemma 3.2. (i) *Linear estimates: Let $T \in (0, 1)$ and $b \in \big(\frac{1}{2}, \frac{3}{2}\big]$. Then, for $s \in \mathbb{R}$ and $\theta \in \big[0, \frac{3}{2} - b\big)$, we have*

$$\|\eta_T(t) S(t)\phi\|_{X^{s,b}(\mathbb{R}\times\mathbb{R}^4)} \lesssim T^{\frac{1}{2}-b}\|\phi\|_{H^s(\mathbb{R}^4)}, \tag{3.7}$$

$$\bigg\|\eta(t)\int_0^t S(t-t')\eta_T(t')F(t')dt'\bigg\|_{X^{s,b}(\mathbb{R}\times\mathbb{R}^4)} \lesssim T^\theta \|F\|_{X^{s,b-1+\theta}(\mathbb{R}\times\mathbb{R}^4)}.$$

(ii) *Strichartz estimates: Let (q,r) be Schrödinger admissible and $p \geq 3$. Then, for $b > \frac{1}{2}$ and $N_1 \leq N_2$, we have*

$$\|u\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^4)} \lesssim \|u\|_{X^{0,b}(\mathbb{R}\times\mathbb{R}^4)}, \tag{3.8}$$

$$\|u\|_{L^p_{t,x}(\mathbb{R}\times\mathbb{R}^4)} \lesssim \big\||\nabla|^{2-\frac{6}{p}}u\big\|_{X^{0,b}(\mathbb{R}\times\mathbb{R}^4)}, \tag{3.9}$$

$$\|\mathbf{P}_{N_1}u_1\mathbf{P}_{N_2}u_2\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^4)} \lesssim N_1\left(\frac{N_1}{N_2}\right)^{\frac{1}{2}}\|\mathbf{P}_{N_1}u_1\|_{X^{0,b}(\mathbb{R}\times\mathbb{R}^4)}\|\mathbf{P}_{N_2}u_2\|_{X^{0,b}(\mathbb{R}\times\mathbb{R}^4)}. \tag{3.10}$$

Recall that (3.8) follows from the Strichartz estimate (1.9) and (3.9) follows from Sobolev inequality and (1.9), while (3.10) follows from a refinement of the Strichartz estimate by Bourgain [9] and Ozawa-Tsutsumi [41].

As a corollary to Lemma 3.2, we have the following estimates.

Lemma 3.3. *Given small $\varepsilon > 0$, let $\varepsilon_1 = 2\varepsilon+$. Then, for $N_1 \leq N_2$, we have*

$$\|u\|_{L^{\frac{3}{1+\varepsilon_1}}_{t,x}(\mathbb{R}\times\mathbb{R}^4)} \lesssim \|u\|_{X^{0,\frac{1}{2}-2\varepsilon}(\mathbb{R}\times\mathbb{R}^4)}, \tag{3.11}$$

$$\|\mathbf{P}_{N_1}u_1\mathbf{P}_{N_2}u_2\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^4)} \lesssim C(N_1,N_2)\|\mathbf{P}_{N_1}u_1\|_{X^{0,\frac{1}{2}+}(\mathbb{R}\times\mathbb{R}^4)}\|\mathbf{P}_{N_2}u_2\|_{X^{0,\frac{1}{2}-2\varepsilon}(\mathbb{R}\times\mathbb{R}^4)}, \tag{3.12}$$

where $C(N_1,N_2)$ is given by

$$C(N_1,N_2) = \begin{cases} N_1^{\frac{3}{2}+\varepsilon_1+}N_2^{-\frac{1}{2}+\varepsilon_1} & \text{if } N_1 \leq N_2,\\ N_1^{-\frac{1}{2}+5\varepsilon_1+}N_2^{\frac{3}{2}-3\varepsilon_1} & \text{if } N_1 \geq N_2.\end{cases}$$

Proof. The first estimate (3.11) follows from interpolating (3.8) with $q = r = 3$ and $\|u\|_{L^2_{t,x}} = \|u\|_{X^{0,0}}$. The second estimate (3.12) follows from interpolating (3.10) and

$$\|\mathbf{P}_{N_1}u_1\mathbf{P}_{N_2}u_2\|_{L^2_{t,x}} \leq \|\mathbf{P}_{N_1}u_1\|_{L^\infty_{t,x}}\|\mathbf{P}_{N_2}u_2\|_{L^2_{t,x}} \lesssim \|\mathbf{P}_{N_1}u_1\|_{X^{2+,\frac{1}{2}+}}\|\mathbf{P}_{N_2}u_2\|_{X^{0,0}}. \qquad \square$$

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. We only prove (3.5) since (3.6) follows in a similar manner. By Lemma 3.2 (i) and duality, we have

$$\begin{aligned}\|\Gamma v(t)\|_{X^{\sigma,b}} &\lesssim T^\theta\|\mathcal{N}(\eta v + \eta_T z)\|_{X^{\sigma,b-1+\theta}}\\ &= T^\theta \sup_{\|v_4\|_{X^{0,1-b-\theta}}\leq 1}\bigg|\iint_{\mathbb{R}\times\mathbb{R}^4}\langle\nabla\rangle^\sigma\big[\mathcal{N}(\eta v+\eta_T z)\big]v_4\,dxdt\bigg|. \end{aligned}\tag{3.13}$$

We estimate the right-hand side of (3.13) by performing case-by-case analysis of expressions of the form:

$$\bigg|\iint_{\mathbb{R}\times\mathbb{R}^4}\langle\nabla\rangle^\sigma(w_1w_2w_3)v_4\,dxdt\bigg|, \tag{3.14}$$

where $\|v_4\|_{X^{0,1-b-\theta}} \leq 1$ and $w_j = \eta v$ or $\eta_T z$, $j = 1,2,3$. Before proceeding further, let us simplify some of the notations. In the following, we drop the complex conjugate sign since it plays no role. Also, we often suppress the smooth cutoff function η (and η_T) from $w_j = \eta v$ (and $w_j = \eta_T z$) and simply denote them by v_j (and z_j, respectively). Lastly, in most of the cases, we dyadically decompose w_j, $j = 1,2,3$, and v_4 such that their spatial frequency supports are $\{|\xi_j| \sim N_j\}$ for some dyadic $N_j \geq 1$ but still denote them by w_j, $j = 1, 2, 3$, and v_4.

Let $b = \frac{1}{2} + \varepsilon$ and $\theta = \varepsilon$ for some small $\varepsilon > 0$ (to be chosen later) so that $1 - b - \theta = \frac{1}{2} - 2\varepsilon$. In the following, we set $\varepsilon_1 = 2\varepsilon+$.

Case (1): vvv case.

In this case, we do not need to perform dyadic decompositions and we divide the frequency spaces into $\{|\xi_1| \geq |\xi_2|, |\xi_3|\}$, $\{|\xi_2| \geq |\xi_1|, |\xi_3|\}$, and $\{|\xi_3| \geq |\xi_1|, |\xi_2|\}$. Without loss of generality, assume that $|\xi_1| \gtrsim |\xi_2|, |\xi_3|$. By $L^3 L^{\frac{6}{1-\varepsilon_1}} L^{\frac{6}{1-\varepsilon_1}} L^{\frac{3}{1+\varepsilon_1}}$-Hölder's inequality and Lemmata 3.2 and 3.3, we have

$$\begin{aligned} \bigg| \int_{\mathbb{R}\times\mathbb{R}^4} \langle \nabla \rangle^\sigma v_1 v_2 v_3 v_4 dx dt \bigg| &\leq \| \langle \nabla \rangle^\sigma v_1 \|_{L^3_{t,x}} \|v_2\|_{L^{\frac{6}{1-\varepsilon_1}}_{t,x}} \|v_3\|_{L^{\frac{6}{1-\varepsilon_1}}_{t,x}} \|v_4\|_{L^{\frac{3}{1+\varepsilon_1}}_{t,x}} \\ &\lesssim \prod_{j=1}^3 \|v_j\|_{X^{\sigma, \frac{1}{2}+}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} \lesssim \prod_{j=1}^3 \|v_j\|_{X^{\sigma,b}} \end{aligned}$$

for $\sigma \geq 1 + \varepsilon_1 = 1 + 2\varepsilon+$.

Case (2): zzz case.

Without loss of generality, assume $N_3 \geq N_2 \geq N_1$.

- **Subcase (2.a):** $N_2 \sim N_3$.

By $L^{\frac{6}{1-2\varepsilon_1}} L^4 L^4 L^{\frac{3}{1+\varepsilon_1}}$-Hölder's inequality and Lemmata 3.2 and 3.3, we have

$$\bigg| \int_{\mathbb{R}\times\mathbb{R}^4} z_1 z_2 \langle \nabla \rangle^\sigma z_3 v_4 dx dt \bigg| \lesssim \|z_1\|_{L^{\frac{6}{1-2\varepsilon_1}}_{t,x}} \|\langle \nabla \rangle^{\frac{\sigma}{2}} z_2\|_{L^4_{t,x}} \|\langle \nabla \rangle^{\frac{\sigma}{2}} z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}.$$

Hence, by Proposition 1.3, the contribution to (3.14) in this case is at most $\lesssim R^3$ outside a set of probability

$$\leq C \exp\bigg(- c \frac{R^2}{T^{\frac{1-2\varepsilon_1}{3}} \|\phi\|^2_{L^2}} \bigg) + C \exp\bigg(- c \frac{R^2}{T^{\frac{1}{2}} \|\phi\|^2_{H^s}} \bigg) \tag{3.15}$$

provided that $s > \frac{\sigma}{2}$. Note that s needs to be strictly greater than $\frac{\sigma}{2}$ due to the summations over dyadic blocks. For the convenience of readers, we briefly show how this follows. In summing $\|\langle \nabla \rangle^{\frac{\sigma}{2}} \mathbf{P}_{N_3} z_3\|_{L^4_{t,x}}$ over dyadic blocks in N_3, we have

$$\begin{aligned} \sum_{\substack{N_3 \geq 1 \\ \text{dyadic}}} \|\langle \nabla \rangle^{\frac{\sigma}{2}} \mathbf{P}_{N_3} z_3\|_{L^4_{t,x}} &\leq \Big(\sum_{N_3} N_3^{0-} \Big)^{\frac{3}{4}} \|\langle \nabla \rangle^{\frac{\sigma}{2}+} \mathbf{P}_{N_3} z_3\|_{\ell^4_{N_3} L^4_{t,x}} \\ &= \Big(\sum_{N_3} N_3^{0-} \Big)^{\frac{3}{4}} \|\langle \nabla \rangle^{\frac{\sigma}{2}+} \mathbf{P}_{N_3} z_3\|_{L^4_{t,x} \ell^4_{N_3}} \\ &\leq \Big(\sum_{N_3} N_3^{0-} \Big)^{\frac{3}{4}} \|\langle \nabla \rangle^{\frac{\sigma}{2}+} \mathbf{P}_{N_3} z_3\|_{L^4_{t,x} \ell^2_{N_3}} \lesssim \|\langle \nabla \rangle^{\frac{\sigma}{2}+} z_3\|_{L^4_{t,x}}, \end{aligned}$$

where the last inequality follows from the Littlewood-Paley theory. By Proposition 1.3 with $q = r = 4$, we obtain the second term in (3.15) as long as $s > \frac{\sigma}{2}$. Moreover, while the terms with z_1 and z_2 also suffer a slight loss of derivative, we can hide the loss in N_1 and N_2 under the z_3-term since $N_3 \geq N_1, N_2$. Similar comments also apply in the sequel.

• **Subcase (2.b):** $N_3 \sim N_4 \gg N_1, N_2$.

◦ Subsubcase (2.b.i): $N_1, N_2 \ll N_3^{\frac{1}{3}}$.

We include the detailed calculation only in this case, with similar comments applicable in the following. By Lemmata 3.2 (ii) and 3.3, with $b = \frac{1}{2}+$ and $\delta = 0+$, we have

$$\begin{aligned}\bigg|\int_{\mathbb{R}\times\mathbb{R}^4} z_1 z_2 \langle\nabla\rangle^\sigma z_3 v_4 dxdt\bigg| &\lesssim \|z_1 \langle\nabla\rangle^\sigma z_3\|_{L^2_{t,x}} \|z_2 v_4\|_{L^2_{t,x}} \\ &\lesssim N_1^{\frac{3}{2}} N_3^{-\frac{1}{2}+\sigma} N_2^{\frac{3}{2}+\varepsilon_1+\delta} N_4^{-\frac{1}{2}+\varepsilon_1} \prod_{j=1}^3 \|z_j\|_{X^{0,b}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} \\ &\lesssim N_1^{\frac{3}{2}-s} N_2^{\frac{3}{2}+\varepsilon_1-s+\delta} N_3^{-\frac{1}{2}+\sigma-s} N_4^{-\frac{1}{2}+\varepsilon_1} \prod_{j=1}^3 \|z_j\|_{X^{s,b}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}\end{aligned}$$

By $N_1, N_2, \ll N_3^{\frac{1}{3}}$, $N_3 \sim N_4$, and Lemma 3.2 (i), we have

$$\ll T^{0-} N_3^{-\frac{5}{3}s+\sigma+\frac{4}{3}\varepsilon_1+\frac{1}{3}\delta} \prod_{j=1}^3 \|\mathbf{P}_{N_j}\phi^\omega\|_{H^s} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}.$$

Here, we lost a small power of T in applying (3.7). Note that such a loss in T can be hidden under T^θ in (3.13) and does not cause a problem. Now, we want the power of the largest frequency N_3 to be strictly negative so that we can sum over dyadic blocks. This requires

$$\frac{5}{3}s > \sigma + \frac{4}{3}\varepsilon_1. \tag{3.16}$$

Provided this condition holds, using Lemma 2.2, we see that the contribution to (3.14) in this case is at most $\lesssim T^{0-}R^3$ outside a set of probability

$$\leq C \exp\bigg(-c\frac{R^2}{\|\phi\|^2_{H^s}}\bigg).$$

◦ Subsubcase (2.b.ii): $N_2 \gtrsim N_3^{\frac{1}{3}} \gg N_1$.

By Lemmata 3.2 and 3.3, we have

$$\begin{aligned}\bigg|\int_{\mathbb{R}\times\mathbb{R}^4} z_1 z_2 \langle\nabla\rangle^\sigma z_3 v_4 dxdt\bigg| &\lesssim \|z_2\|_{L^4_{t,x}} \|\langle\nabla\rangle^\sigma z_3\|_{L^4_{t,x}} \|z_1 v_4\|_{L^2_{t,x}} \\ &\lesssim T^{0-} N_1^{\frac{3}{2}+\varepsilon_1-s+} N_2^{-s} N_3^{\sigma-s-\frac{1}{2}+\varepsilon_1} \|\mathbf{P}_{N_1}\phi^\omega\|_{H^s} \prod_{j=2}^3 \|\langle\nabla\rangle^s z_j\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}.\end{aligned}$$

Hence, by Lemma 2.2 and Proposition 1.3, the contribution to (3.14) in this case is at most $\lesssim T^{0-}R^3$ outside a set of probability

$$\leq C \exp\bigg(-c\frac{R^2}{\|\phi\|^2_{H^s}}\bigg) + C \exp\bigg(-c\frac{R^2}{T^{\frac{1}{2}}\|\phi\|^2_{H^s}}\bigg)$$

provided that (3.16) is satisfied.

∘ <u>Subsubcase (2.b.iii):</u> $N_1, N_2 \geq N_3^{\frac{1}{3}}$.

By $L^{\frac{9}{2-\varepsilon_1}} L^{\frac{9}{2-\varepsilon_1}} L^{\frac{9}{2-\varepsilon_1}} L^{\frac{3}{1+\varepsilon_1}}$-Hölder's inequality and Lemmata 3.2 and 3.3, we have

$$\bigg| \int_{\mathbb{R}\times\mathbb{R}^4} z_1 z_2 \langle\nabla\rangle^\sigma z_3 v_4 dx dt \bigg| \lesssim N_3^{\sigma-\frac{5}{3}s} \prod_{j=1}^{3} \big\| \langle\nabla\rangle^s z_j \big\|_{L_{t,x}^{\frac{9}{2-\varepsilon_1}}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}.$$

Hence, by Proposition 1.3, the contribution to (3.14) in this case is at most $\lesssim R^3$ outside a set of probability

$$\leq C \exp\bigg(-c \frac{R^2}{T^{\frac{4-2\varepsilon_1}{9}} \|\phi\|_{H^s}^2} \bigg)$$

provided that

$$\frac{5}{3}s > \sigma. \tag{3.17}$$

Therefore, given $s > \frac{3}{5}$, we choose $\sigma = 1+$ and $\varepsilon = 0+$ for Case (2) such that (3.16) and (3.17) are satisfied.

Case (3): vvz case.

Without loss of generality, assume $N_1 \geq N_2$.

- **Subcase (3.a):** $N_1 \gtrsim N_3$.

By $L^3 L^{\frac{6}{1-\varepsilon_1}} L^{\frac{6}{1-\varepsilon_1}} L^{\frac{3}{1+\varepsilon_1}}$-Hölder's inequality and Lemmata 3.2 and 3.3, we have

$$\bigg| \int_{\mathbb{R}\times\mathbb{R}^4} \langle\nabla\rangle^\sigma v_1 v_2 z_3 v_4 dx dt \bigg| \lesssim \|v_1\|_{X^{\sigma,\frac{1}{2}+}} \|v_2\|_{X^{1+\varepsilon_1,\frac{1}{2}+}} \|z_3\|_{L_{t,x}^{\frac{6}{1-\varepsilon_1}}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}.$$

Hence, by Proposition 1.3, the contribution to (3.14) in this case is at most $\lesssim R \prod_{j=1}^2 \|v_j\|_{X^{\sigma,\frac{1}{2}+}}$ outside a set of probability

$$\leq C \exp\bigg(-c \frac{R^2}{T^{\frac{1-\varepsilon_1}{3}} \|\phi\|_{H^{0+}}^2} \bigg) \tag{3.18}$$

provided that $\sigma > 1 + \varepsilon_1 = 1 + 2\varepsilon+$. Note that we have $\|\phi\|_{H^{0+}}$ instead of $\|\phi\|_{L^2}$ in (3.18) due to the summation over N_3.

- **Subcase (3.b):** $N_3 \sim N_4 \gg N_1 \geq N_2$.

By Lemmata 3.2 and 3.3, we have

$$\begin{aligned}
\bigg| \int_{\mathbb{R}\times\mathbb{R}^4} v_1 v_2 \langle\nabla\rangle^\sigma z_3 v_4 dx dt \bigg| &\lesssim \|v_1\|_{L^4_{t,x}} \|\langle\nabla\rangle^\sigma z_3\|_{L^4_{t,x}} \|v_2 v_4\|_{L^2_{t,x}} \\
&\lesssim N_2^{\frac{3}{2}+\varepsilon_1-\sigma+} N_3^{\sigma-s} N_4^{-\frac{1}{2}+\varepsilon_1} \|v_1\|_{X^{\frac{1}{2},\frac{1}{2}+}} \|v_2\|_{X^{\sigma,\frac{1}{2}+}} \|\langle\nabla\rangle^s z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} \\
&\lesssim N_1^{2-2\sigma+\varepsilon_1+} N_3^{\sigma-s-\frac{1}{2}+\varepsilon_1} \|v_1\|_{X^{\sigma,\frac{1}{2}+}} \|v_2\|_{X^{\sigma,\frac{1}{2}+}} \|\langle\nabla\rangle^s z_3\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}.
\end{aligned}$$

Hence, by Proposition 1.3, the contribution to (3.14) in this case is at most $\lesssim R \prod_{j=1}^2 \|v_j\|_{X^{\sigma,\frac{1}{2}+}}$ outside a set of probability

$$\leq C \exp\bigg(-c \frac{R^2}{T^{\frac{1}{2}} \|\phi\|_{H^s}^2} \bigg)$$

provided that $2-2\sigma+\varepsilon_1<0$ and $s>\sigma-\frac{1}{2}+\varepsilon_1$. Given $s>\frac{1}{2}$, these conditions are satisfied by taking $\sigma=1+$ and $\varepsilon=0+$.

Case (4): *vzz* case.

Without loss of generality, assume $N_3\geq N_2$.

- **Subcase (4.a):** $N_1\gtrsim N_3$.

By $L^3L^{\frac{6}{1-\varepsilon_1}}L^{\frac{6}{1-\varepsilon_1}}L^{\frac{3}{1+\varepsilon_1}}$-Hölder's inequality and Lemmata 3.2 and 3.3, we have

$$\bigg|\int_{\mathbb{R}\times\mathbb{R}^4}\langle\nabla\rangle^\sigma v_1z_2z_3v_4dxdt\bigg|\lesssim\|v_1\|_{X^{\sigma,\frac{1}{2}+}}\|z_2\|_{L^{\frac{6}{1-\varepsilon_1}}_{t,x}}\|z_3\|_{L^{\frac{6}{1-\varepsilon_1}}_{t,x}}\|v_4\|_{X^{0,\frac{1}{2}-\varepsilon}}.$$

Hence, by Proposition 1.3, the contribution to (3.14) in this case is at most $\lesssim R^2\|v_1\|_{X^{\sigma,\frac{1}{2}+}}$ outside a set of probability

$$\leq C\exp\bigg(-c\frac{R^2}{T^{\frac{1-\varepsilon_1}{3}}\|\phi\|^2_{H^{0+}}}\bigg).$$

- **Subcase (4.b):** $N_3\gg N_1$.

First, suppose that $N_2\sim N_3$. Then, by Lemmata 3.2 and 3.3 (after separating the argument into two cases: $N_1\leq N_4$ or $N_1\geq N_4$), we have

$$\begin{aligned}\bigg|\int_{\mathbb{R}\times\mathbb{R}^4}v_1z_2\langle\nabla\rangle^\sigma z_3v_4dxdt\bigg|&\lesssim\|\langle\nabla\rangle^{\frac{\sigma}{2}}z_2\|_{L^4_{t,x}}\|\langle\nabla\rangle^{\frac{\sigma}{2}}z_3\|_{L^4_{t,x}}\|v_1v_4\|_{L^2_{t,x}}\\&\lesssim N_1^{1+2\varepsilon_1-\sigma+}N_3^{\sigma-2s}\|v_1\|_{X^{\sigma,\frac{1}{2}+}}\|\langle\nabla\rangle^sz_2\|_{L^4_{t,x}}\|\langle\nabla\rangle^sz_3\|_{L^4_{t,x}}\|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}.\end{aligned}$$

Hence, by Proposition 1.3, the contribution to (3.14) in this case is at most $\lesssim R^2\|v_1\|_{X^{\sigma,\frac{1}{2}+}}$ outside a set of probability

$$\leq C\exp\bigg(-c\frac{R^2}{T^{\frac{1}{2}}\|\phi\|^2_{H^s}}\bigg)$$

provided that $\sigma>1+2\varepsilon_1$ and $s>\frac{1}{2}\sigma$. Given $s>\frac{1}{2}$, these conditions are satisfied by taking $\sigma=1+$ and $\varepsilon=0+$.

Hence, it remains to consider the case $N_3\sim N_4\gg N_1,N_2$.

∘ <u>Subsubcase (4.b.i):</u> $N_1,N_2\ll N_3^{\frac{1}{3}}$.

By Lemmata 3.2 and 3.3, we have

$$\begin{aligned}\bigg|\int_{\mathbb{R}\times\mathbb{R}^4}v_1z_2\langle\nabla\rangle^\sigma z_3v_4dxdt\bigg|&\lesssim\|v_1\langle\nabla\rangle^\sigma z_3\|_{L^2_{t,x}}\|z_2v_4\|_{L^2_{t,x}}\\&\lesssim T^{0-}N_1^{\frac{3}{2}-\sigma}N_2^{\frac{3}{2}+\varepsilon_1-s+}N_3^{\sigma-s-\frac{1}{2}}N_4^{-\frac{1}{2}+\varepsilon_1}\|v_1\|_{X^{\sigma,\frac{1}{2}+}}\prod_{j=2}^3\|\mathbf{P}_{N_j}\phi^\omega\|_{H^s}\|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}\\&\lesssim T^{0-}N_3^{\frac{2}{3}\sigma-\frac{4}{3}s+\frac{4}{3}\varepsilon_1+}\|v_1\|_{X^{\sigma,\frac{1}{2}+}}\prod_{j=2}^3\|\mathbf{P}_{N_j}\phi^\omega\|_{H^s}\|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}.\end{aligned}$$

Hence, by Lemma 2.2, the contribution to (3.14) in this case is at most $\lesssim T^{0-}R^2\|v_1\|_{X^{\sigma,\frac{1}{2}+}}$ outside a set of probability

$$\leq C\exp\bigg(-c\frac{R^2}{\|\phi\|^2_{H^s}}\bigg)$$

provided that

$$s > \frac{1}{2}\sigma + \varepsilon_1. \tag{3.19}$$

Given $s > \frac{1}{2}$, this condition is satisfied by taking $\sigma = 1+$ and $\varepsilon = 0+$.

∘ <u>Subsubcase (4.b.ii):</u> $N_1 \ll N_3^{\frac{1}{3}} \lesssim N_2$.
By Lemmata 3.2 and 3.3, we have

$$\begin{aligned}\bigg|\int_{\mathbb{R}\times\mathbb{R}^4} v_1 z_2 \langle\nabla\rangle^\sigma z_3 v_4 dxdt\bigg| &\lesssim \|z_2\|_{L^4_{t,x}} \|\langle\nabla\rangle^\sigma z_3\|_{L^4_{t,x}} \|v_1 v_4\|_{L^2_{t,x}} \\ &\lesssim N_1^{\frac{3}{2}+\varepsilon_1-\sigma+} N_2^{-s} N_3^{\sigma-s-\frac{1}{2}+\varepsilon_1} \|v_1\|_{X^{\sigma,\frac{1}{2}+}} \prod_{j=2}^3 \|\langle\nabla\rangle^s z_j\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} \\ &\lesssim N_3^{\frac{2}{3}\sigma-\frac{4}{3}s+\frac{4}{3}\varepsilon_1+} \|v_1\|_{X^{\sigma,\frac{1}{2}+}} \prod_{j=2}^3 \|\langle\nabla\rangle^s z_j\|_{L^4_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}.\end{aligned}$$

Hence, by Proposition 1.3, the contribution to (3.14) in this case is at most $\lesssim R^2\|v_1\|_{X^{\sigma,\frac{1}{2}+}}$ outside a set of probability

$$\le C\exp\bigg(-c\frac{R^2}{T^{\frac{1}{2}}\|\phi\|_{H^s}^2}\bigg)$$

provided that (3.19) is satisfied.

∘ <u>Subsubcase (4.b.iii):</u> $N_2 \ll N_3^{\frac{1}{3}} \lesssim N_1$.
By Lemmata 3.2 and 3.3, we have

$$\begin{aligned}\bigg|\int_{\mathbb{R}\times\mathbb{R}^4} v_1 z_2 \langle\nabla\rangle^\sigma z_3 v_4 dxdt\bigg| &\lesssim \|v_1\|_{L^3_{t,x}} \|\langle\nabla\rangle^\sigma z_3\|_{L^6_{t,x}} \|z_2 v_4\|_{L^2_{t,x}} \\ &\lesssim T^{0-} N_1^{-\sigma} N_2^{\frac{3}{2}+\varepsilon_1-s+} N_3^{\sigma-s-\frac{1}{2}+\varepsilon_1} \|v_1\|_{X^{\sigma,\frac{1}{2}+}} \|\mathbf{P}_{N_2}\phi^\omega\|_{H^s} \|\langle\nabla\rangle^s z_3\|_{L^6_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} \\ &\lesssim T^{0-} N_3^{\frac{2}{3}\sigma-\frac{4}{3}s+\frac{4}{3}\varepsilon_1+} \|v_1\|_{X^{\sigma,\frac{1}{2}+}} \|\mathbf{P}_{N_2}\phi^\omega\|_{H^s} \|\langle\nabla\rangle^s z_3\|_{L^6_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}.\end{aligned}$$

Hence, by Lemma 2.2 and Proposition 1.3, the contribution to (3.14) in this case is at most $\lesssim T^{0-}R^2\|v_1\|_{X^{\sigma,\frac{1}{2}+}}$ outside a set of probability

$$\le C\exp\bigg(-c\frac{R^2}{\|\phi\|_{H^s}^2}\bigg) + C\exp\bigg(-c\frac{R^2}{T^{\frac{1}{3}}\|\phi\|_{H^s}^2}\bigg)$$

provided that (3.19) is satisfied.

∘ <u>Subsubcase (4.b.iv):</u> $N_1, N_2 \gtrsim N_3^{\frac{1}{3}}$.

By $L^3 L^{\frac{6}{1-\varepsilon_1}} L^{\frac{6}{1-\varepsilon_1}} L^{\frac{3}{1+\varepsilon_1}}$-Hölder's inequality and Lemmata 3.2 and 3.3, we have

$$\begin{aligned}\bigg|\int_{\mathbb{R}\times\mathbb{R}^4} v_1 z_2 \langle\nabla\rangle^\sigma z_3 v_4 dxdt\bigg| &\lesssim \|v_1\|_{L^3_{t,x}} \|z_2\|_{L^{\frac{6}{1-\varepsilon_1}}_{t,x}} \|\langle\nabla\rangle^\sigma z_3\|_{L^{\frac{6}{1-\varepsilon_1}}_{t,x}} \|v_4\|_{L^{\frac{3}{1+\varepsilon_1}}_{t,x}} \\ &\lesssim N_1^{-\sigma} N_2^{-s} N_3^{\sigma-s} \|v_1\|_{X^{\sigma,\frac{1}{2}+}} \prod_{j=2}^3 \|\langle\nabla\rangle^s z_j\|_{L^{\frac{6}{1-\varepsilon_1}}_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}} \\ &\lesssim N_3^{\frac{2}{3}\sigma-\frac{4}{3}s} \|v_1\|_{X^{\sigma,\frac{1}{2}+}} \prod_{j=2}^3 \|\langle\nabla\rangle^s z_j\|_{L^{\frac{6}{1-\varepsilon_1}}_{t,x}} \|v_4\|_{X^{0,\frac{1}{2}-2\varepsilon}}.\end{aligned}$$

Hence, by Proposition 1.3, the contribution to (3.14) in this case is at most $\lesssim R^2 \|v_1\|_{X^{\sigma,\frac{1}{2}+}}$ outside a set of probability

$$\leq C \exp\bigg(-c\frac{R^2}{T^{\frac{1-\varepsilon_1}{3}}\|\phi\|^2_{H^s}}\bigg)$$

provided that $s > \frac{1}{2}\sigma$. Given $s > \frac{1}{2}$, this condition is satisfied by setting $\sigma = 1+$. □

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